Technical Notes

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Improper Integrals in Theoretical Aerodynamics: The Problem Revisited

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Introduction

THE term *improper integral* is used to designate integrals having one or more infinite limits of integration or having a finite domain of integration with internal singularities. This Note will discuss integrals of the latter type.

An integral of the type

$$\int_a^b \frac{f(x)\mathrm{d}x}{(x_0-x)^\alpha}$$

where $f(x)$ is a bounded function in the interval and $a \le x_0 \le b$, converges only if $\alpha < 1$. If $0<\alpha<1$, the singularity is said to be of the "weak" type. This integral diverges if $\alpha \ge 1$, in which case the singularity is said to be of the "strong" type. For integrals of the latter kind, Hadamard[1] has established the concept of "finite part" as an extension of Cauchy's principal value of single-pole integrals. His idea will be applied in the following sections.

Mangler[2] and Lighthill[3] have given formulas for the computation of the finite part of some integrals that occur frequently in aerodynamics. For the case of poles of integer order, Mangler's formula reads as follows:

$$\mathrm{FP}\int_a^b \frac{f(x)\mathrm{d}x}{(x_0-x)^{n+1}} = \lim_{\epsilon\to 0}\left\{\int_a^{x_0-\epsilon} \frac{f(x)\mathrm{d}x}{(x_0-x)^{n+1}} + \int_{x_0+\epsilon}^b \frac{f(x)\mathrm{d}x}{(x_0-x)^{n+1}} + (-1)^n \sum_{j=0}^{n-1} \frac{f^{(j)}(x_0)}{j!}\,\frac{[1-(-1)^{n-j}]}{(n-j)\epsilon^{n-j}}\right\} \tag{1}$$

where $\epsilon > 0$, n is a non-negative integer, $a < x_0 < b$, and the superscript j of the function f denotes the jth derivative of this function with respect to x. Lighthill's equivalent result for the case $x_0 = 0$ is given by

$$\mathrm{FP}\int_a^b \frac{f(x)}{x^{n+1}}\,\mathrm{d}x = \frac{1}{n!}\lim_{\epsilon\to 0}\left\{\int_a^{-\epsilon} \frac{f^{(n)}(x)}{x}\,\mathrm{d}x + \int_\epsilon^b \frac{f^{(n)}(x)}{x}\,\mathrm{d}x\right\} - \sum_{j=1}^{n} \frac{(j-1)!}{n!}\left[\frac{f^{(n-j)}(b)}{b^j} - \frac{f^{(n-j)}(a)}{a^j}\right] \tag{2}$$

While Mangler's technique relies mainly on breaking up the integration interval into regularly integrable regions and adding correction terms, Lighthill's method is based solely on the technique of integration by parts. However, both formulas are similar in nature, as each requires the evaluation of numerical limits. This fact, among others, inspired this author to search for a direct way of solving the problem.

An Alternate Approach

Let us consider the problem of evaluating the finite part of the following integral:

$$\int_a^b \frac{f(x)\mathrm{d}x}{(x_0-x)^{\alpha+n+1}}$$

where $a \le x_0 \le b$, $0<\alpha<1$, and $n \ge -1$. We will assume that the function $f(x)$ is bounded in the interval $a \le x \le b$ and has all necessary derivatives well defined and finite at the singularity x_0. With these hypotheses, we can use part of the Taylor series expansion of $f(x)$ about $x = x_0$ to write the following:

$$\mathrm{FP}\int_a^b \frac{f(x)\mathrm{d}x}{(x_0-x)^{\alpha+n+1}} = \int_a^b \left[f(x) + \sum_{j=0}^{n+1} \frac{(-1)^{j+1}}{j!} f^{(j)}(x_0)(x_0-x)^j\right]\frac{\mathrm{d}x}{(x_0-x)^{\alpha+n+1}} - \sum_{j=0}^{n+1}\frac{(-1)^{j+1}}{j!} f^{(j)}(x_0)\left[\mathrm{FP}\int_a^b \frac{\mathrm{d}x}{(x_0-x)^{\alpha+n+1-j}}\right] \tag{3a}$$

For the case of integrals with poles of integer order ($\alpha = 0$), we need one less derivative and Eq. (3a) reduces to

$$\mathrm{FP}\int_a^b \frac{f(x)\mathrm{d}x}{(x_0-x)^{n+1}} = \int_a^b \left[f(x) + \sum_{j=0}^{n} \frac{(-1)^{j+1}}{j!} f^{(j)}(x_0)(x_0-x)^j\right]\frac{\mathrm{d}x}{(x_0-x)^{n+1}} - \sum_{j=0}^{n}\frac{(-1)^{j+1}}{j!} f^{(j)}(x_0)\left[\mathrm{FP}\int_a^b \frac{\mathrm{d}x}{(x_0-x)^{n+1-j}}\right] \tag{3b}$$

In these two equations, the finite part of a divergent integral has been transformed into the sum of a regular integral and simpler finite-part integrals. We will call these latter integrals the *finite-part integrals of elementary poles*. The regular integral is designed to be evaluated numerically as a single piece. The terms added to and subtracted from the original problem are those required to regularize the integrand at the singularity in the sense of L'Hôpital. This idea works like the *canonical regularization* of Gel'fand and Shilov[4] or Kanwal.[5]

The present approach has three advantages over previous methods. First, the regularized integral has a continuous integrand that can be handled by ordinary integration schemes. Second, the finite part of the problem is reduced to analytical work, with improvements in accuracy. Finally, no

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numerical limit is required. The answer is obtained directly, with considerable savings in computing time.

The Finite-Part Integral of Elementary Poles

Here we will derive formulas for the exact computation of the finite-part integral of elementary poles. Let us start by considering the problem

$$\mathfrak{F} = \mathrm{FP}\int_a^b \frac{\mathrm{d}x}{(x_0-x)^{\alpha+n+1}}$$

where $a<x_0<b$. Following Hadamard,[1] the answer is given by

$$\mathfrak{F} = \lim_{\epsilon\to 0}\left\{\int_a^{x_0-\epsilon}\frac{\mathrm{d}x}{(x_0-x)^{\alpha+n+1}} + \int_{x_0+\epsilon}^b \frac{\mathrm{d}x}{(x_0-x)^{\alpha+n+1}} + G(\epsilon)\right\} \tag{4}$$

where ϵ is a small positive real quantity and $G(\epsilon)$ is the function of ϵ canceling out *exactly* the terms resulting from the integrals that are singular in the limit.

The first integral in Eq. (4) yields

$$\int_a^{x_0-\epsilon}\frac{\mathrm{d}x}{(x_0-x)^{\alpha+n+1}} = -\frac{1}{(\alpha+n)}\left[\frac{1}{(x_0-a)^{\alpha+n}} - \frac{1}{\epsilon^{\alpha+n}}\right]$$

In the interval $(x_0+\epsilon)\le x\le b$, the quantity (x_0-x) is negative. So, we can write

$$(x_0-x) = (-1)(x-x_0) = e^{-i\pi}(x-x_0)$$

Using y as a dummy variable to denote $(x-x_0)$, the second integral in Eq. (4) can be worked out as follows:

$$\int_{x_0+\epsilon}^b \frac{\mathrm{d}x}{(x_0-x)^{\alpha+n+1}} = \Re\left[e^{i(\alpha+n+1)\pi}\right]\int_\epsilon^{b-x_0}\frac{\mathrm{d}y}{y^{\alpha+n+1}}$$

$$= -\frac{\cos(\alpha+n+1)\pi}{(\alpha+n)}\left[\frac{1}{(b-x_0)^{\alpha+n}} - \frac{1}{\epsilon^{\alpha+n}}\right]$$

where $\Re$ denotes the real part of what comes between square brackets.

Adding the previous results and choosing the function $G(\epsilon)$ to be equal to

$$G(\epsilon) = -\frac{1+\cos(\alpha+n+1)\pi}{(\alpha+n)\epsilon^{\alpha+n}}$$

the limit in Eq. (4) becomes independent of ϵ and is given by

$$\mathfrak{F} = -\frac{(b-x_0)^{\alpha+n} + (x_0-a)^{\alpha+n}\cos(\alpha+n+1)\pi}{(\alpha+n)\left[(x_0-a)(b-x_0)\right]^{\alpha+n}} \tag{5}$$

For the particular case of poles of integer order, Eq. (5) reduces to

$$\mathrm{FP}\int_a^b \frac{\mathrm{d}x}{(x_0-x)^{n+1}} = -\frac{(b-x_0)^n + (-1)^{n+1}(x_0-a)^n}{n\left[(x_0-a)(b-x_0)\right]^n} \tag{6}$$

This is not valid when $n=0$. For this case, the result is given by

$$\mathrm{FP}\int_a^b \frac{\mathrm{d}x}{(x_0-x)} = \ln\frac{(x_0-a)}{(b-x_0)} \tag{6a}$$

Now let us deal with the case where the singularity is placed either on the lower or upper limit of integration. In the former case, the answer comes from

$$\mathrm{FP}\int_a^b \frac{\mathrm{d}x}{(a-x)^{\alpha+n+1}} = \lim_{\epsilon\to 0}\left\{\int_{a+\epsilon}^b \frac{\mathrm{d}x}{(a-x)^{\alpha+n+1}} + G(\epsilon)\right\} \tag{7}$$

The integral here can be worked out as follows:

$$\int_{a+\epsilon}^b \frac{\mathrm{d}x}{(a-x)^{\alpha+n+1}} = \Re\left[e^{i(\alpha+n+1)\pi}\right]\int_\epsilon^{b-a}\frac{\mathrm{d}y}{y^{\alpha+n+1}}$$

$$= -\frac{\cos(\alpha+n+1)\pi}{(\alpha+n)}\left[\frac{1}{(b-a)^{\alpha+n}} - \frac{1}{\epsilon^{\alpha+n}}\right]$$

where y is the dummy variable for $(x-a)$. Choosing the corresponding function $G(\epsilon)$ as given by

$$G(\epsilon) = -\frac{\cos(\alpha+n+1)\pi}{(\alpha+n)\epsilon^{\alpha+n}}$$

Eq. (7) assumes the following explicit form:

$$\mathrm{FP}\int_a^b \frac{\mathrm{d}x}{(a-x)^{\alpha+n+1}} = -\frac{\cos(\alpha+n+1)\pi}{(\alpha+n)(b-a)^{\alpha+n}} \tag{8}$$

When $\alpha=0$, Eq. (8) reduces to

$$\mathrm{FP}\int_a^b \frac{\mathrm{d}x}{(a-x)^{n+1}} = \frac{(-1)^n}{n(b-a)^n} \tag{9a}$$

Again, this formula cannot be applied when $n=0$. In this case, we have

$$\mathrm{FP}\int_a^b \frac{\mathrm{d}x}{(a-x)} = -\ln(b-a) \tag{9b}$$

Similar development can be done with respect to singularities at the upper limit of integration.

Some Extensions to the Present Method

Usually we can find situations where the integrand has a finite number of internal singularities located at m different places within the interval, limits included. Equations (3) can be generalized even further to accommodate this situation. The most general result we can present now is

$$\mathrm{FP}\int_a^b f(x)\left[\prod_{i=1}^m (x_{0_i}-x)^{\alpha_i+n_i+1}\right]^{-1}\mathrm{d}x$$

$$= \int_a^b \left\{f(x) + \sum_{i=1}^m\left[\sum_{j=0}^{n_i+1}\frac{(-1)^{j+1}}{j!}f^{(j)}(x_{0_i})(x_{0_i}-x)^j\right]\right.$$

$$\left.\times\prod_{k=1,k\neq i}^m\left(\frac{x_{0_k}-x}{x_{0_k}-x_{0_i}}\right)^{n_k+2}\right\}\left[\prod_{i=1}^m (x_{0_i}-x)^{\alpha_i+n_i+1}\right]^{-1}\mathrm{d}x$$

$$-\sum_{i=1}^m\left\{\sum_{j=0}^{n_i+1}\frac{(-1)^{j+1}}{j!}f^{(j)}(x_{0_i})\left[\prod_{k=1,k\neq i}^m (x_{0_k}-x_{0_i})^{n_k+2}\right]^{-1}\right.$$

$$\left.\times\mathrm{FP}\int_a^b (x_{0_i}-x)^{-(\alpha_i+n_i+1-j)}\prod_{k=1,k\neq i}^m (x_{0_k}-x)^{1-\alpha_k}\mathrm{d}x\right\} \tag{10}$$

When a singularity has $\alpha=0$, we should replace n by $n-1$ on the right-hand side, except when it occurs as exponent of

Table 1 Present method using Gauss-Legendre quadrature

n	2	4	8	16
$\mathfrak{F}$	−4.76832	−4.76803	−4.76803	−4.76803

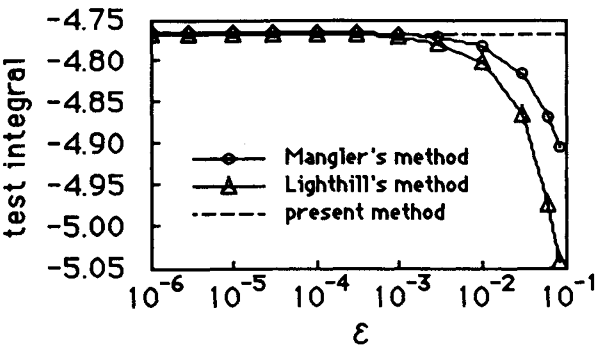

Fig. 1 Convergence of previous methods for the test integral.

$(x_{0_i}-x)$. Also, we should make the binomial $(x_{0_k}-x)$ inside the finite-part integrals equal to 1. Although extra work will be required for the evaluation of these integrals, a solution can be obtained by breaking the interval into pieces between singularities and following a derivation similar to those presented here.

In two or more dimensions, the methods of Mangler or Lighthill lead to results that depend on the geometry of the "hole" built around the singularity. This difficulty has led researchers to call these integrals "semi-convergent." The procedure being shown here eliminates the concept of isolating the singularity and brings uniqueness to the treatment of singular integrals. Therefore, it should be considered an effective method of handling this kind of problem and one which will help in the understanding of singular integral equations.

An Example

To allow a direct comparison between the three integration methods under discussion, let us consider the following test problem and its associated answers:

$$\mathfrak{F} = \mathrm{FP}\int_{-1}^{1} \frac{e^x \mathrm{d}x}{(0.5-x)^2}$$

1) Mangler's method

$$\mathfrak{F} = \lim_{\epsilon\to 0}\left\{\int_{-1}^{0.5-\epsilon} \frac{e^x \mathrm{d}x}{(0.5-x)^2} + \int_{0.5+\epsilon}^{1} \frac{e^x \mathrm{d}x}{(0.5-x)^2} - \frac{2e^{0.5}}{\epsilon}\right\} \tag{11}$$

2) Lighthill's method

$$\mathfrak{F} = -\lim_{\epsilon\to 0}\left\{\int_{-0.5}^{-\epsilon} \frac{e^{0.5-x}}{x}\,\mathrm{d}x + \int_{\epsilon}^{1.5} \frac{e^{0.5-x}}{x}\,\mathrm{d}x\right\} - \frac{2}{3e} - 2e \tag{12}$$

3) Present method

$$\mathfrak{F} = \int_{-1}^{1} \frac{e^x - (0.5+x)e^{0.5}}{(0.5-x)^2}\,\mathrm{d}x - \left(\frac{8}{3} + \ln 3\right)e^{0.5} \tag{13}$$

The equations above look quite different from each other. However, as shown in Fig. 1, they provide virtually identical results. For values of ϵ smaller than 10^{-3}, all three results seem to agree rather well with the exact answer (−4.7680302), which can be obtained in terms of exponential integrals.[6] The important point, however, is how these answers are obtained. For the integrals in Eqs. (11) and (12), we have used an algorithm proposed by Stoer and Bulirsch[7] to approach the singularity accordingly. However, the smaller the value of ϵ is, the more computing time is required by the methods of Mangler or Lighthill.

In contrast, Table 1 shows outcomes of the present approach. For the integration we have used Gauss-Legendre quadrature formulas[8] of order n. As shown, a simple four-point rule has been sufficient to freeze a result with six significant figures. Savings in computing time can be described as greater than two orders of magnitude.

Conclusions

A direct method for the evaluation of the finite part of divergent integrals has been presented. This technique has its roots on a pioneering work by Hadamard[1] and has been reviewed by Lomax, Heaslet and Fuller.[9] Compared to the methods of Mangler[2] or Lighthill,[3] the present one is simpler. Its main advantages are the elimination of the numerical limit that appears in both previous approaches and the reduction of the finite part of the problem to pure analytical work. This yields faster and more accurate numerical results. Therefore, the present scheme is expected to increase significantly the numerical efficiency of large codes which involve these kinds of integral.

Formulas for the finite part integral of some elementary poles have been derived. They cover a larger range of applications than those previously available in the literature. Finally, a double-pole test integral has been computed by all methods to highlight the advantages of the present approach.

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